Universality in passively advected hydrodynamic fields: the case of a passive vector with pressure.

R. Benzi[1], L. Biferale[1] and F. Toschi[1,2]
[1] Dipartimento di Fisica and INFM, Universita' di Roma "Tor Vergata", Via della Ricerca Scientifica 1, I-00133 Roma, Italy
[2] University of Twente, Department of Applied Physics, Enschede, The Netherlands

Universality of statistical properties of passive quantities advected by turbulent velocity fields at changing the passive forcing mechanism is discussed. In particular, we concentrate on the statistical properties of an hydrodynamic system with pressure. We present theoretical arguments and preliminary numerical results which show that the fluxes of passive vector field and of the velocity field have the same scaling behavior. By exploiting such a property, we propose a way to compute the anomalous exponents of three dimensional turbulent velocity fields. Our findings are in agreement within 5% with experimental values of the anomalous exponents.

PACS: 47.27-i

I. INTRODUCTION

Recent theoretical and numerical studies of passive quantities linearly advected by either stochastic [1,2,10,4,18] or true turbulent Navier-Stokes velocity fields [5] have focused on the important problem of the "zero-mode" dominance of the statistical properties of the passive field in the inertial range.
The field may be an advected scalar $\theta(\boldsymbol{x}, t)$, with the equation of motion

$$\frac{\partial \theta}{\partial t} + \boldsymbol{u} \cdot \nabla \theta = \kappa \nabla^2 \theta + f_\theta, \tag{1}$$

or a vector, like a magnetic field $\boldsymbol{B}(\boldsymbol{x}, t)$ satisfying [7]

$$\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{B} = \boldsymbol{B} \cdot \nabla \boldsymbol{u} + \kappa \nabla^2 \boldsymbol{B} \; + \mathbf{f}_B. \tag{2}$$

Zero-mode dominance is of crucial importance because naturally explains the deviations from dimensional predictions for scaling properties and some universality with respect to the forcing mechanism, i.e. some independence of scaling properties from the large scale input of passive field.

In particular, for the special class of Kraichnan-like problems for scalar and vectors with Gaussian and white-in-time velocity field and external forcing [1,2,10,4], one can prove that the equal time (passive) correlation functions are dominated, in the inertial range, by the zero modes of the linear operator describing the advection by the Gaussian velocity field. Anomalous scaling comes from the non-trivial scaling properties of the null-space of the inertial operator. Universality comes from the fact that *only* prefactor of the zero modes feel the large scale boundary conditions, while the power law behavior is fixed by the inertial operator. As a consequence of zero-mode dominance in the inertial range we also have that all forcing-dependent contributions are sub-dominant and therefore some degree of universality with respect to the forcing mechanism.

Recently, numerical evidences that zero modes show up also in true passive models, advected by realistic non-Gaussian and non white-in-time velocity fields, have been presented for both passive scalars in the inverse cascade regime of a two dimensional flow [5] and shell models for passive scalars [6]. Such findings give strong support to the idea that statistics of quantities linearly advected by true Navier-Stokes fields may show some universality properties with respect to the forcing mechanism too. In particular, the problem of a passive with a mean shear studied in [5] shows that zero-modes remain dominant with respect to the forcing contributions also in presence of a correlation between forcing and velocity advecting field, i.e. some degree of universality still holds. In this paper we want to investigate how far one can push the idea of universality with respect to the forcing mechanism for the particular case of the advection of a passive field with pressure [18] advected by the true Navier-Stokes turbulent velocity field. Such a model was initially proposed in [18] in order to understand the importance of non-local contributions, induced by the pressure term, in the zero-modes structure of the advecting operator. Here, we want to show and study another important striking feature of the model, i.e. its property to describe either a *linear* or *non-linear* evolution depending on the correlation between the passive forcing and the advecting velocity fields. Moreover, when the passive evolution is indeed non-linear, the passive field and the velocity fields coincide, i.e. the transported field and the advecting field

are looked in both their spatial and temporal evolution. In other words, we have a linear model which may become non-linear by changing the correlation properties between the external forcing and the advecting velocity field.

In this paper we address two problems. First, we want to understand the degree of universality in the scaling properties of the *linear* problem at changing the external forcing.
Second, by using the similarity between the non-linear Navier-Stokes eqs and the linear advecting-diffusive eqs for the passive field (see below) we propose a way to compute the anomalous exponents of homogeneous and isotropic turbulence which takes advantage of recent results in the theory of passive scalar advected by Gaussian, white in time velocity fields [1,2].
The paper is organized as follows. In section II we present the equations for the passive field with pressure. in section III we present some numerical experiments made on the equivalent shell model for the passive field and we discuss to which extent we may expect strong universality with respect to the forcing mechanism in this model. In particular we discuss why we expect, and verify numerically in the shell model case, that passive fluxes posses a much better degree of universality than passive structure functions. In section IV we used the, supposed, universality of passive flux in order to derive a functional constraint for the velocity statistics. Conclusions follow in section V.

II. THE MODEL

Let us consider the incompressible Navier-Stokes (NS) equations:

$$\partial_t U_i + U_j \partial_j U_i = -\partial_i p + \nu \Delta U_i + f_i \tag{3}$$

where notation follows the usual meaning and the forcing term f_i is supposed to produce a stationary homogeneous and isotropic turbulence. We next consider a vector field $\boldsymbol{W}$, divergence-less (i.e. $\partial_i W_i = 0$) and satisfying a transport-like equation:

$$\partial_t W_i + U_j \partial_j W_i = -\partial_i p_w + \nu \Delta W_i + g_i \tag{4}$$

where the "pressure" term p_w is computed via the Poisson equation:

$$\partial_i U_j \partial_j W_i = -\Delta p_w \tag{5}$$

Our aim is to discuss the statistical properties of $\boldsymbol{W}$ and their relation with the scaling properties of $\boldsymbol{U}$.

Let us introduce the anomalous exponents for $\boldsymbol{U}$ and $\boldsymbol{W}$. We shall denote by $\zeta(p)$ the scaling exponents of the longitudinal structure function:

$$S^p(r) = \langle (\delta_r U)^p \rangle \sim r^{\zeta(p)}$$

where $\delta_r U = (\boldsymbol{U}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{U}(\boldsymbol{x})) \cdot \hat{\boldsymbol{r}}$ and $\langle \cdot \rangle$ stands for ensemble averaging. In the same way, we introduce $\sigma(p)$ as the scaling exponents of longitudinal linear-field structure functions:

$$T^p(r) = \langle (\delta_r W)^p \rangle \sim r^{\sigma(p)}$$

where $\delta_r W = (\boldsymbol{W}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{W}(\boldsymbol{x})) \cdot \hat{\boldsymbol{r}}$. Finally,we consider the scaling exponents $s(p)$ for the $\boldsymbol{W}$ *flux* defined as:

$$\langle (\delta_r U)^p (\delta_r W)^{2p} \rangle \sim r^{s(p)} \tag{6}$$

Let us remark that equation (4) implies $s(1) = 0$ which follows from the analogous of the "4/5" Kolmogorov equation for $\boldsymbol{W}$ [3], namely

$$\langle \delta_r U (\delta_r W)^2 \rangle \sim N r \tag{7}$$

where N is the mean rate of W dissipation. Our analysis is aimed at understanding the relationship, if any, among the anomalous exponents $\zeta(p), \sigma(p), s(p)$. We are now able to pose our problem in a quantitative way. Let us first consider the very simple case $f_i = g_i$. By subtracting equation (3) from equation (4) we obtain:

$$\partial_t \phi_i + U_j \partial_j \phi_i = -\partial_i \pi + \nu \Delta \phi_i \tag{8}$$

where $\boldsymbol{\phi} = \boldsymbol{W} - \boldsymbol{U}$ and $\pi = (p - p_w)$. By equation (8) it immediately follow that the space average E_ϕ of $\boldsymbol{\phi}^2 = \int dx \phi_i(x)\phi_i(x)$ satisfies the equation:

$$\partial_t E_\phi = -\epsilon_\phi \tag{9}$$

where ϵ_ϕ is the mean rate of dissipation of E_ϕ. Thus, for long enough time, the field ϕ_i goes to zero and $W_i = U_i$, identically.

In this paper we want to understand to which extent the statistical properties of (4) are universal with respect to the forcing mechanism. If a strongly universality holds, then we should have independently of the forcing mechanism

$$\sigma(p) = \zeta(p), \quad s(p) = \zeta(3p) \tag{10}$$

even when f_i and g_i are uncorrelated or weakly correlated fields. Let us notice that in the previous equality we have assumed that velocity flux possesses the same scaling properties of the velocity field, as always verified in all numerical and experimental data. We do not know any rigorous argument against or in favor of universality for the scaling properties of (4). One may argue that if the forcing mechanisms f_i and g_i are weakly correlated –or independent– and confined only to large scales than the statistical properties of $\boldsymbol{W}$ may not be strongly influenced by the forcing itself. In the latter case the universality of the scaling exponents of $\boldsymbol{W}$ should be achieved by the same zero-modes mechanisms previously discussed for the passive scalar with independent forcing. Whether this supposed universality can be pushed until the very extreme case of fully correlated systems, implying the equalities (10) is a matter of discussion. There is clearly a physical relevant question here: we need to understand the importance of external forcing mechanism in the statistical properties of the advected field. This question, of course, arises not only for the passive vector with pressure but also in any linear advection problem, i.e. passive scalars or passive magnetic fields. In the case of the passive vector with pressure the question assumes also another interest because of the possibility to push the advected field to follow the velocity field for some particular external forcing.

We consider an important point to investigate this problem in details and in the next section, we present some numerical results showing that universality holds for fluxes, i.e. $s(p) = \zeta(3p)$. Passive structure functions seems to be less universal than fluxes although strong boundary effects (both ultraviolet and infrared) do not allow a precise statement.

III. NUMERICAL RESULTS

Although a direct numerical tests of the statistical properties (4) is possible for the NS equations, we will limit in this paper to a direct numerical investigation in the framework of shell models for three dimensional turbulence.
Shell models for turbulent energy cascade have proved to share many statistical properties with both turbulent three dimensional velocity fields [3,12,14] and with passive linearly advected quantities [13]. Let us introduces a set of wavenumber $k_n = \lambda^n k_0$ with $n = 0, \ldots, N$ and the inter-shell ratio fixed to $\lambda = 2$. The shell-velocity variables $u_n(t)$ must be understood as the velocity fluctuation over a distances $r_n = k_n^{-1}$. We also introduce the linear-field shell variables $w_n(t)$ defined on the same sets of discrete wavenumber. It is possible to write down a set of coupled ODEs for the time evolution of $u_n(t)$ and $w_n(t)$ which mimics the velocity and the passive turbulent evolution (see below).

We know that zero modes are at works in shell models for linearly advected quantities, exactly like in the true linear hydrodynamical problems. Indeed, it is possible to prove analytically that shell models for passive scalar advected by Gaussian and white-in-time velocity (shell) fields have intermittent corrections dominated by the null space of the linear finite-dimensional advecting operator [9,11]. Recently, it has also been shown that shell models for passive scalar advection, i.e. passive shells advected by shell fields arising from a shell model for the velocity, have zero modes dominance for the scaling properties in the inertial shells, exactly as it is shown for the true passive scalar [5].
We have therefore, exactly the analytical/phenomenological framework useful to check to which extent the scaling properties of the linear hydrodinamical problem are forcing independent.

In the shell model framework the very meaning of pressure is absent. The equivalent of our linear-hydrodynamical model (4) will become a shell field $w_n(t)$, linearly advected by the shell model velocity fields $u_n(t)$, conserving the energy $\sum_n |w_n|^2$, and such that when $f_n = g_n$ we have, for time large enough, $w_n(t) = u_n(t)$.

It is possible to write down such a coupled set of ODEs for all shell models. Here we consider the case of Sabra model [14]. We obtain:

$$(\frac{d}{dt} + \nu k_n^2) u_n = i(k_n u_{n+1}^* u_{n+2} + b k_{n-1} u_{n+1} u_{n-1}^* + (1+b) k_{n-2} u_{n-2} u_{n-1}) + f_n \tag{11}$$

$$(\frac{d}{dt}+\nu k_n^2)w_n = i(k_n u_{n+1}^* w_{n+2} + b k_{n-1} w_{n+1} u_{n-1}^* + k_{n-2} w_{n-2} u_{n-1} + b k_{n-2} w_{n-1} u_{n-2}) + g_n \tag{12}$$

where the non linear term of u_n evolution and the linear advection part of w_n evolution have the only free parameter b. Note that if we put $w_n = u_n$ in equation (12), then equation (12) becomes equivalent to equation (11). It is known that in order to have a realistic intermittent behavior for u_n one has to chose $-1 < b < 0$ [14].
Equations (11) and (12) preserve velocity energy, $E_u \equiv \sum_n |u_n|^2$, and passive energy, $E_w \equiv \sum_n |w_n|^2$, in the limit of zero viscosity and zero forcing.

It is easy to show that if $f_n = g_n$ in the above equations than for long enough time $w_n(t) \equiv u_n(t)$.

The universality issue, as discussed in the previous section, consists now in studying the scaling properties of w_n at changing its forcing mechanism.
Let us define the scaling exponents for the velocity flux,

$$\Pi_n = \Im[(k_n u_{n+2} u_{n+1}^* u_n^*) + (1+b) k_{n-1}(u_{n+1} u_n^* u_{j-1}^*)]$$

and passive flux,

$$Q_n = \Im[(k_n w_{n+2} u_{n+1}^* w_n^*) + k_{n-1}(w_{n+1} u_n^* w_{n-1}^*) + k_{n-1} b (w_{n+1} w_n^* u_{n-1}^*)]$$

as:

$$S_\Pi^p(n) \equiv \langle |\Pi_n|^p \rangle \sim k_n^{p-\zeta(3p)} \quad T_Q^p(n) \equiv \langle |Q_n|^p \rangle \sim k_n^{p-s(p)} \tag{13}$$

where the equivalent of 4/5 law for the two shell models gives $\zeta(3) = s(1) = 1$, [15]. The structure functions are defined as follows:

$$S_u^p(n) = \langle |u_n|^p \rangle \sim k_n^{-\zeta(p)} \quad T_w^p(n) = \langle |w_n|^p \rangle \sim k_n^{-\sigma(p)} \tag{14}$$

In the following we present some numerical tests done with $N = 25$ shells $\nu_u = \nu_w = 5 \cdot 10^{-7}$ and different kind of forcing mechanisms.
Forcing have been chosen such as to go from a weakly correlated situation where $g_n(t)$ has some large scale dependency from the $u_n(t)$ dynamics to a fully uncorrelated case with $g_n(t)$ given by a random process.

In Fig. 1 we present the typical scaling laws one obtains for fluxes of both fields, for different large scale passive-forcing. In all simulations we have always taken the same velocity forcing concentrated on the largest shell and constant: $f_n = (1+i)C_u \delta_{n,1}$, with $C_u = 0.01$. We have compared statistical properties of the passive fields using three different choices for the passive forcing. In case (A) we had a time-dependent forcing such as to impose a constant passive energy input on the first shell:

$$A: \quad g_n(t) = \delta_{n,1}(1+i)/w_1^*(t).$$

In case (B) we fixed the first passive shell to have the same amplitude $|w_1| = const.$ but leaving its phase to evolve according to its own dynamics.

$$B: \quad g_n(t) \to |w_1(t)| = 1. \ \forall t.$$

In case (C) we took a forcing concentrated on the first shell, with constant amplitude, $|g_1(t)| = G_1$ but random independent phases

$$C: \quad g_n(t) = \delta_{n,1} G_1 e^{i\theta(t)}$$

where $\langle \theta(t)\theta(t') \rangle \propto \delta(t-t')$.
Let us notice that forcing of cases (A) and (B) have some (weak) correlation with the advecting velocity field, while case (C) is independent of u_n.
As one can see all flux curves superpose perfectly in the inertial range. Let us notice that the extremely small errors on the scaling of two fluxes allows us to support the statement $\Pi_n \sim Q_n$ with very high accuracy independently on the forcing mechanism. The scaling of passive structure functions suffers of larger error bars, due to a less smooth matching between inertial and infra-red properties [16]. The qualitative trend is, anyhow, toward a more intermittent behavior of the passive fields with respect to the velocity field (see Fig. 2). In the latter case we would have an indication that passive structure functions are much more sensible to the boundary conditions than the fluxes moments, implying also a weaker degree of universality with respect to the forcing mechanism.

In Table 1 we quantify our finding in all three forcing cases (A-C) showing the best fits and their errors for all scaling exponents of both fluxes and structure functions.
In Fig. 3, we show two typical time evolution for the total energies of velocity, $E_u(t)$, and passive, $E_w(t)$ for the case with two different -but correlated- forcings, $f_n \neq g_n$. Although some correlation between $E_u(t)$ and $E_w(t)$ is observed, we are very far from the trivial exact correlated case one would have obtained choosing $f_n \equiv g_n$. The non trivial correlations between the two energy can be seen in the fig. 4 where we plot, for a typical trajectory, E_u versus E_w.
In order to test the robustness of previous results we repeated the numerical experiments with a different value of the free parameter b in the sabra shell model equations (11). It is known that at changing b the intermittency of the velocity field changes.
In Fig. 5 we plot the $\zeta(3p)$ and $s(p)$ curves for both values of $b = -0.4$ and $b = -0.6$. The agreement for each b values between the two fluxes is again perfect despite the fact that at varying b we have different degrees of intermittency.

One could question why there is such clear evidence that $s(p) = \zeta(3p)$, i.e. the scaling properties of two fluxes are identical, while passive structure functions seem to be more intermittent than the corresponding velocity structure functions. One possible explanation goes as follows. By exploiting the equation of motion one may always derives homogeneous constraints for moments of quantities like $F_{\boldsymbol{n}} = u_n w_{n'} w_{n''}$ in the inertial range, i.e. where forcing is not directly acting. In particular by writing the stationary condition for quantities like $\langle F_{\boldsymbol{n}_1} F_{\boldsymbol{n}_2} \cdots F_{n_{p-1}} w_k w_{k'} \rangle$ one obtains an homogeneous constraints involving only $F_{\boldsymbol{n}}$ observable:

$$\frac{d}{dt}\langle F_{\boldsymbol{n}_1} F_{\boldsymbol{n}_2} \cdots F_{\boldsymbol{n}_{p-1}} w_k w_{k'} \rangle = O^{\boldsymbol{m}_1 \boldsymbol{m}_2 \cdots \boldsymbol{m}_p}_{\boldsymbol{n}_1 \boldsymbol{n}_2 \cdots \boldsymbol{n}_{p-1},k,k'} \langle F_{\boldsymbol{m}_1} F_{\boldsymbol{m}_2} \cdots F_{\boldsymbol{m}_p} \rangle = 0 \tag{15}$$

where the operator $O^{\boldsymbol{m}_1 \boldsymbol{m}_2 \cdots \boldsymbol{m}_p}_{\boldsymbol{n}_1 \boldsymbol{n}_2 \cdots \boldsymbol{n}_{p-1},k,k'}$ is given from the equation of motion and it is independent of the correlation between u_n and w_n and of the chosen forcings g_n and f_n. On the other hands no homogeneous closed constraints can ever be found for moments of quantities involving only simultaneous correlations of passive fields, w_n. In order to have homogeneous expressions for obervable made of only passive fields one has to give (or to guess) an explicit form for the correlation between w_n and u_n. Thus one may think to obtain an equation similar to (15) but now with the inertial operator explicitly dependent on the given correlation between the two fields. In the latter case the dependency of the passive field on the statistical properties of the velocity field and of the correlation with the external forcing may lead to non-universal scaling properties.
On the other hand, one can also argue that the observed lack of universality is not Reynolds independent and that at Reynolds large enough some strong independence from the forcing mechanism (zero-modes dominance) would be recovered also for passive structure functions. The latter scenario is what happens in true passive scalars with a correlation between forcing and velocity field (see the case of a passive scalar with shear [8]) where the existence of sub-leading non-homogeneous terms induced by the forcing mechanism may spoil the scaling behavior of the zero-modes at finite Reynolds numbers.

IV. AN APPROXIMATE COMPUTATION OF THE SCALING EXPONENTS

The results so far discussed, make us rather confident that the two fluxes of equation (4) and of Navier Stokes eqs (3) have the same statistical fluctuations.

We want now to understand whether it is possible to use the above results to obtain useful informations on the scaling exponents $\zeta(p)$.

The main idea, discussed in this section, is the following. We assume that the velocity field can be described by the (unknown) multifractal probability distribution $P(\delta_r U)$. We want to compute the probability distribution of the passive vector, $P(\delta_r W)$. The equation (4) will induce a functional relation between the two probability distributions. By requiring that the scaling properties of $\boldsymbol{W}$ flux are the same of $\boldsymbol{U}$ flux, we introduce an (infinite) set of equations for the probability $P(\delta_r U)$ whose solutions will fix its scaling. In order to simplify the above procedure, the approach we want to follow is based on a suitable set of approximations and assumptions whose validity we are not able to prove rigorously, only *a posteriori* we will be able to judge the goodness of our calculation.
Let us suppose that the field $\boldsymbol{U}$ in equation (4) is characterized by a multifractal spectrum $D(h)$. As it is well known, in such a case the anomalous exponents of $\boldsymbol{U}$ are given by the expression $\zeta(p) = \inf_h (ph + 3 - D(h))$ and the probability distribution of $\delta_r U$ is given by [17]:

$$P(\delta_r U) \propto \int d\mu(h) r^{3-D(h)} \exp\left[-\frac{(\delta_r U)^2}{2U_0 r^{2h}}\right] \tag{16}$$

where U_0 represents the variance of the large scale velocity field, which is supposed to be Gaussian. Equation (16) tells us that for each value of r we may consider $\delta_r U$ as the superposition of Gaussian field with variance $U_0 r^{2h}$ and probability $r^{3-D(h)}$ associated to each value of h. We can then consider to solve equation (4) in the limit where $\boldsymbol{U}$ is a "weighted" superposition of Gaussian random fields (as expressed by (16)).

The equivalence between the statistical properties of the Navier-Stokes field and of the linear field implies that both fluxes have the same scaling:

$$\langle |(\delta_r U)(\delta_r W)^2|^p \rangle \sim \langle |\delta_r U|^{3p} \rangle \tag{17}$$

We want to exploit the above identity in order to derive a constraint for the $D(h)$ spectrum.
We can rewrite the term $\langle (\delta_r U(\delta_r W)^2)^p \rangle$ as follows:

$$\langle (\delta_r U(\delta_r W)^2)^p \rangle \sim \int dh d(\delta_r U) r^{3-D(h)} P(\delta_r U, h)(\delta_r U)^p \langle (\delta_r W)^{2p} | \delta_r U, h \rangle \tag{18}$$

where with $\langle (\delta_r W)^{2p} | \delta_r U, h \rangle$ we mean the average of the linear field conditioned to the advecting velocity field and $P(\delta_r U, h) \equiv \exp\left[-\frac{(\delta_r U)^2}{2U_0 r^{2h}}\right]$. Up to eqn. (18) we did not used any approximation. The computation of $\langle (\delta_r W)^{2p} | \delta_r U, h \rangle$ is the most difficult one and we shall introduce several approximations in order to make it feasible. In particular we want to compute $\langle (\delta_r W)^{2p} | \delta_r U, h \rangle$ using a surrogate δ-correlated and Gaussian velocity field. In other words will make the first approximation that in order to compute the conditional average we may assume that the multifractal velocity field, as defined by equation (16), is the "weighted" superposition of independent Gaussian random fields δ -correlated in time.

In order to compute left hand side of equation (18), we solve equation (4) for fixed exponent h of the random field $\boldsymbol{U}$ and then average the results over h with probability $r^{3-D(h)}$. In doing such an average we notice that in order to mimic the advection by a true Navier-Stokes field (not δ-correlated in time) with scaling $\delta_r U \sim r^h$ we need a δ-correlated surrogate with scaling $\delta_r U_s \sim r^{\frac{1+h}{2}}$, because of dimensions carried by the delta-functions. As a consequence we are going to be interested only in exponents for the surrogate U_s field in the range $H = \frac{1+h}{2} = [1/2, 1]$, which correspond to the range $h : [0, 1]$ for the true turbulent field. Let us notice that a K41 field has a correspondent δ-correlated field scaling as $\delta_r U_s \sim r^{2/3}$.

We now want to exploit our δ-correlated ansatz by writing:

$$\langle (\delta_r W)^{2p} | \delta_r U, h \rangle \sim r^{p(1-h)+\rho_{2p}(1+h)} \tag{19}$$

where we have introduced the anomaly $\rho_{2p}(2H)$ of the linear field $\boldsymbol{W}$ advected by a δ-correlated velocity field $\delta_r U_s \sim r^H$, namely:

$$\langle (\delta_r W)^{2p}) \rangle_H \sim r^{p(2-2H)+\rho_{2p}(2H)} \tag{20}$$

In equation (20) we have taken into account the dimensional consistency relation $2H = 1 + h$. Notice that for the δ-correlated case one can prove that in the isotropic sector we have $\rho_2(2H) = 0$, i.e. the second order correlation function has not anomalous scaling.
Our definition gives:

$$\langle \delta_r U(\delta_r W)^2 \rangle_H \sim r^H r^{1-H} = r \tag{21}$$

i.e, our ansatz on the function $\langle (\delta_r W)^{2p} | \delta_r U, h \rangle$ implies that we are averaging over all possible singularity H with the constraint $\langle \delta_r U(\delta_r W)^2 \rangle_H \sim r$.
We can now use the relation of statistical identity between fluxes, $\zeta(3p) = s(p)$, in order to obtain an equation for $D(h)$. We have:

$$\int d\mu(h) r^{ph+p(1-h)-\rho_{2p}(1+h)+3-D(h)} \sim r^{\zeta(3p)} \tag{22}$$

which gives:

$$\inf_h [p - \rho_{2p}(1+h) + 3 - D(h)] = \inf_h [3ph + 3 - D(h)] \tag{23}$$

If we are able to compute the anomalous correction ρ_{2p} (of the δ-correlated problem), equation (23) becomes a functional equation for $D(h)$ whose solution gives us the anomalous exponents $\zeta(p)$ within the set of approximation discussed above. Let us notice that (23) is consistent with the known result $\tau(1) = s(1) = 0$. Let us also remark that as soon as we introduce any smoothing in the time dependency of $\boldsymbol{U}$ the equation (23) may be no longer valid and we should reconsider the averaging procedure in a more suitable way.
The computation of the anomalous exponents ρ_{2p} is a feasible but difficult task for equation (4) because of the condition $\partial_i W_i = 0$. In principle, the computation can be done perturbatively following the analog of the passive scalar case [2]. Only the exponents in different anisotropic sectors for the second order correlation function have been, up to now, computed. However, in order to understand the quality of the approximations so far introduced to derive equation (23), at least for three dimensional isotropic and homogeneous turbulence, we can use the numerical values of $\rho_4(1+h)$ and $\rho_6(1+h)$ as recently computed numerically for the passive scalar model [19], hoping that the introduction of the pressure term does not change too much the scaling exponents [20].

In order to find a solution of (23), we simply assume that $D(h)$ can be parameterized by using the expression $D(h) = d_0(1 - x + xln(x))$ where $x = (h - h_0)/(d_0 log(\beta^{-1}))$ which corresponds to a log-poison distribution for the multifractal model. We use a log-poison formula because we know that it parametrize particularly well the experimental data [21,22]. Because d_0 is fixed by the condition $\zeta(3) = 1$, we are left with the problem to find β and h_0 by using equation (23) for $p = 2$ and $p = 3$. Solving equation (23) for β and h_0 gives $\beta = 0.781$ and $h_0 = 0.14$. In fig. 6 we show the estimated values of $\zeta(p)$ for $p = 1, ..12$ against the experimental findings for homogeneous and isotropic turbulence. We note that the estimate based on (23) is rather accurate with an error not greater than 5%. We argue that such a small error shows that our approach, within the limitation and the approximations previously discussed, looks promising as an useful tool to compute the anomalous exponents.

In order to test the sensitivity of (23) to the values of $\rho_4(1+h)$ and $\rho_6(1+h)$, it is interesting to use the anomalous exponents ρ_4 and ρ_6 given by Kraichnan' formula for $D = 3$. In this case the solution of (23) gives $\beta = 0.813$ and $h_0 = 0.21$ and the corresponding values of $\zeta(p)$ are also plotted in fig. 6. Although there are not big differences between exponents obtained using the Kraichnan formula and those obtained by using the correct numerical results, the agreement with the experimental data is definitely better in the latter case.

V. CONCLUSIONS

In this paper we discussed several properties characterizing the statistical behavior of a divergence-less vector field passively advected by an homogeneous and isotropic turbulent field. We can summarize our findings in the following way. (i) We present some theoretical arguments which supports the statement that the flux of passive vector field should have the same statistical properties of the flux of an homogeneous and isotropic turbulent field. (ii) We generalize the concept of divergence-less vector field to shell models and by using detailed numerical simulations we provide evidence that the anomalous scaling exponents of the passive vector flux and of the non linear shell model flux are the same with very high accuracy. (iii) We propose a self consistent approach to compute the anomalous exponents in homogeneous and isotropic turbulence by using the properties previously discussed. In particular, we have been able to define a functional equation for $D(h)$ within a suitable set of approximations. (iv) By assuming that the pressure term does not change dramatically the numerical values of the anomalous exponents for the divergence-less vector field advected by a Gaussian isotropic δ-correlated random field, we have been able to find an approximate solution of the functional equation for $D(h)$ which compares rather well to known experimental data.

Before closing we want to discuss a generalization of the model (4) which we consider interesting for further studies. Let us consider the following set of equations:

$$\partial_t U_i + U_j \nabla_j U_i + \lambda W_j \nabla_j U_i = -\nabla_i p_u + \nu \Delta U_i + f_i \tag{24}$$

$$\partial_t W_i + U_j \nabla_j W_i + \lambda W_j \nabla_j W_i = -\nabla_i p_w + \nu \Delta W_i + g_i \tag{25}$$

where we assumed that $\partial_i U_i = \partial_i W_i = 0$. It is interesting to notice that the vector field $\boldsymbol{Z}_\lambda = \boldsymbol{U} + \lambda \boldsymbol{W}$ satisfies the Navier Stokes equations and that equations (24) and (25) corresponds to (3) and (4) for $\lambda = 0$.

Finally we want to remark that equation (4) is an useful tool to understand the role played by coherent structure on the anomalous scaling in both two dimensional and three dimensional turbulence. All our findings suggest that a systematic study of equation (4) looks extremely promising in order to derive a new approach in understanding intermittency and estimating anomalous scaling in Navier-Stokes equations.
We acknowledge useful discussion with I. Arad, A. Celani and M. Vergassola. We thank M. Vergassola and A. Mazzino

who provided us the numerical data for the 4th and 6th order anomaly in the Kraichnan passive scalar problem. This work has been partially supported by the EU under the Grant No. HPRN-CT 2000-00162 “Non Ideal Turbulence” and by the INFM (Iniziativa di Calcolo Parallelo).

---

[1] R.H. Kraichnan, Phys. Rev. Lett. **72** 1016 (1994).
[2] K. Gawedzki and A. Kupiainen, Phys. Rev. Lett. **75** 3834 (1995).
[3] U. Frisch, *Turbulence: the legacy of A.N. Kolmogorov* (Cambridge University Press, Cambridge, England, 1995).
[4] I. Arad, L. Biferale and I. Procaccia ; Phys. Rev. E **61**, 2654, (2000).
[5] A. Celani and M. Vergassola,Phys. Rev. Lett. **86**, 424 (2001).
[6] I. Arad, L. Biferale, A. Celani, I. Procaccia and M. Vergassola, in preparation (2001).
[7] Ya. B. Zeldovich, A. A. Ruzmaikin and D. D. Sokoloff, *Magnetic Fields in Astrophysics* Gordon and Breach Publ. (1983).
[8] A. Celani, A. Lanotte, A.Mazzino, M. Vergassola, “Fronts in passive scalar turbulence”, *Phys. Fluids* in presss (2001); nlin.CD/0009014;
[9] A. Wirth and L. Biferale Phys. Rev. E **54** 4982 (1996).
[10] M. Vergassola, Phys. Rev. E **53** R3021 (1996).
[11] R. Benzi, L. Biferale and A. Wirth, Phys. Rev. Lett. **78** 4926 (1997).
[12] T. Bohr, M.H. Jensen, G. Paladin and A. Vulpiani *Dynamical System Approach to Turbulence* (Cambridge University Press, Cambridge, 1998).
[13] M.H. Jensen, G. Paladin and A. Vulpiani, Phys. Rev. A **43** 798 (1991).
[14] V.S. L’vov, E. Podivilov, A. Pomyalov, I. Procaccia and D. Vandembroucq, Phys. Rev. E **58** 1811 (1998).
[15] D. Pisarenko, L. Biferale, D. Courvoisier, U. Frisch and M. Vergassola ; Phys. Fluids A **5**, 2533, (1993).
[16] Such a different behavior of matching between structure functions and fluxes is common to all shell models. It is meanly due to the fact that the “true” scaling observable here are only fluxes.
[17] R. Benzi, L. Biferale, G. Paladin, A. Vulpiani and M. Vergassola ; Phys. Rev. Lett. **67**, 2299, (1991).
[18] I. Arad and I. Procaccia, “Spectrum of Anisotropic Exponents in Hydrodynamic Systems with Pressure” nlin.CD/0011026 (2000).
[19] U. Frisch, A. Mazzino and M. Vergassola, Phys. Rev. Lett. **80** 5532 (1998).
[20] Let us stress again that this is exactly true for the second order correlation function.
[21] M. Briscolini, P. Santangelo, S. Succi and R. Benzi, Phys. Rev. E **50** R1745 (1994).
[22] R. Benzi, S. Ciliberto, C. Baudet, G. Ruiz Chavarria, Phisica D **80** 385 (1995).

p	$k_n^{-p/3} S_\Pi^{p/3}(n)$	$k_n^{-p/3} T_Q^{p/3}(x)$ (A)	$k_n^{-p/3} T_Q^{p/3}(n)$ (B)	$k_n^{-p/3} T_Q^{p/3}(n)$ (C)	$T_w^p(n)$ (A)	$T_w^p(n)$ (B)	$T_w^p(n)$ (C)
2	0.712 (3)	0.711 (3)	0.711 (2)	0.710 (2)	0.67 (3)	0.67 (3)	0.66 (3)
4	1.263 (6)	1.264 (7)	1.264 (6)	1.266 (3)	1.20 (6)	1.28 (4)	1.13 (7)
6	1.745 (8)	1.741 (7)	1.74 (1)	1.745 (5)	1.6 (1)	1.67 (8)	1.4 (1)
8	2.18 (2)	2.18 (2)	2.19 (2)	2.18 (2)	2.0 (2)	2.1 (1)	1.6 (2)
10	2.60 (2)	2.58 (3)	2.61 (2)	2.57 (4)	2.3 (2)	2.6 (2)	1.9 (3)

TABLE I. Scaling exponents of both flux and structure function of the passive field and of flux of velocity field at changing the order of the moment $p = 2, 4, 6, 8, 10$ and for different large scale forcing cases (A-C). Notice the high precision in the agreement between the passive flux at changing the large-scale forcing (columns 1–4). Passive structure functions (columns 5–7) are less accurate due to the presence of a strong bottleneck at small scales. Errors are given by the numbers in brackets and refers to the last digit. Errors are estimated from the fluctuations of the local logarithmic derivatives in the shell range $n = 3, \dots, 15$.

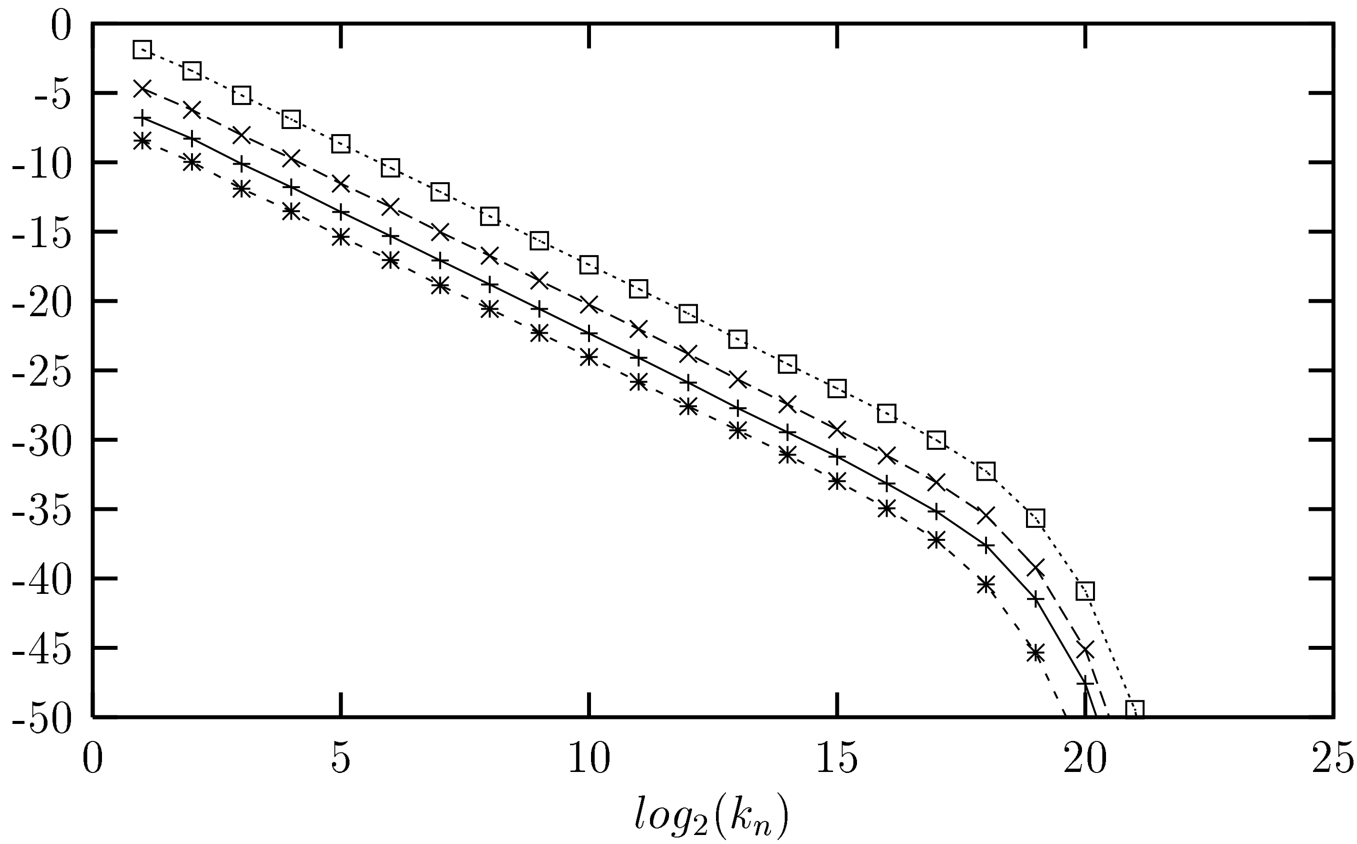

FIG. 1. Log-log plot of velocity and passive fluxes and structure functions for second order moment, $p = 2$ versus the scale k_n. Curves represent from above: ($\Box$) velocity flux $S^2_\Pi(n)$. Passive flux, $T^2_Q(n)$, for forcing case (A) ($\times$); forcing case (B) ($+$); forcing case (C) ($*$). Fluxes are always multiplied by the normalising factor k_n^{-2}. Curves have been shifted along the y-axis for the sake of clarity

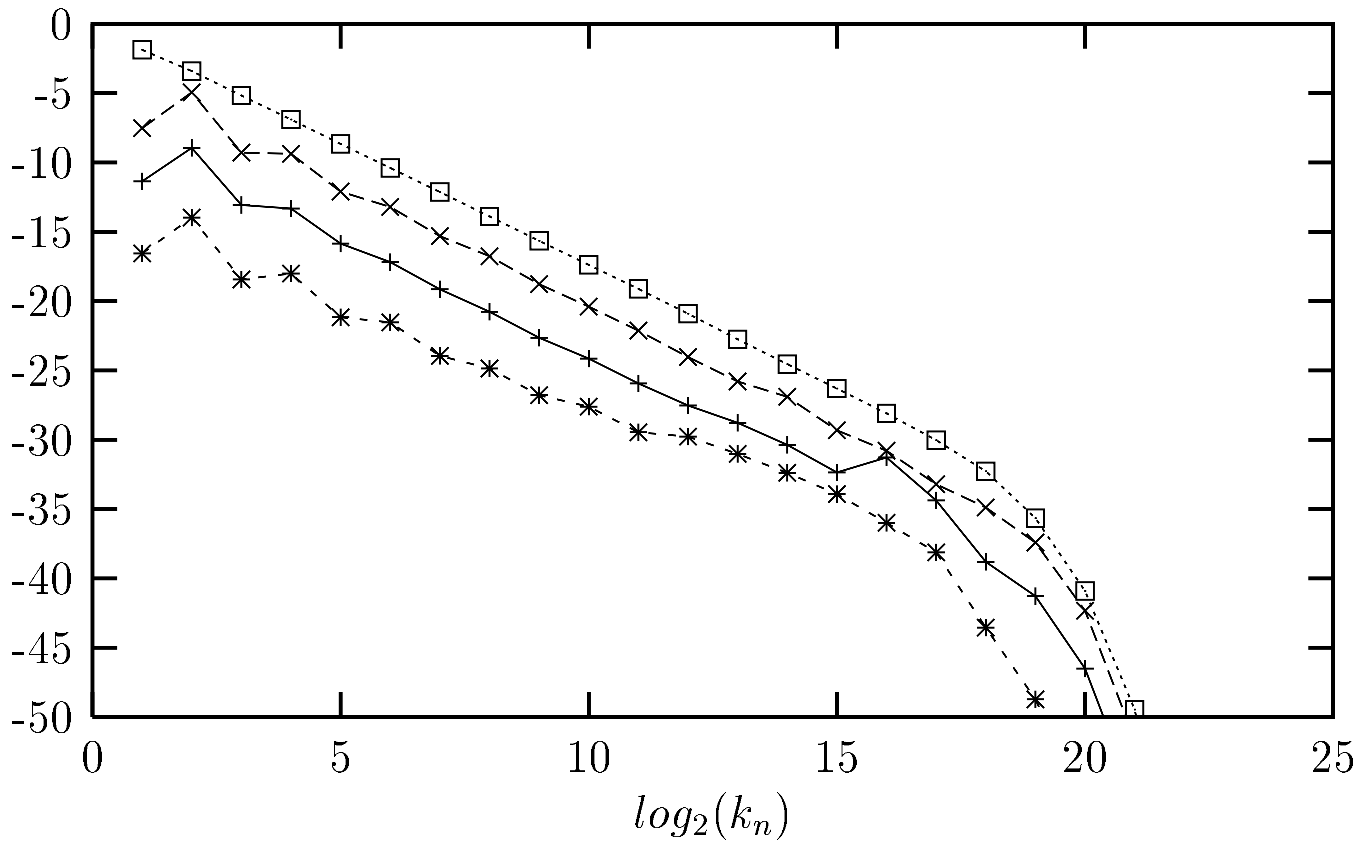

FIG. 2. Log-log plot of second order velocity flux, $p = 2$, and sixth order passive structure functions, versus the scale k_n. Curves represent from above: ($\square$) velocity flux $S_\Pi^2(n)$; ($\times$) passive structure function $T_w^6(n)$ forcing case (A); ($+$) passive structure function $T_w^6(n)$ forcing case (B); ($*$) passive structure function $T_w^6(n)$ forcing case (C). Notice the strong ultraviolet and infrared effects in the passive structure functions. Notice also that for the fully independent forcing, case (C), the passive structure functions shows a larger intermittent slope at small scales.

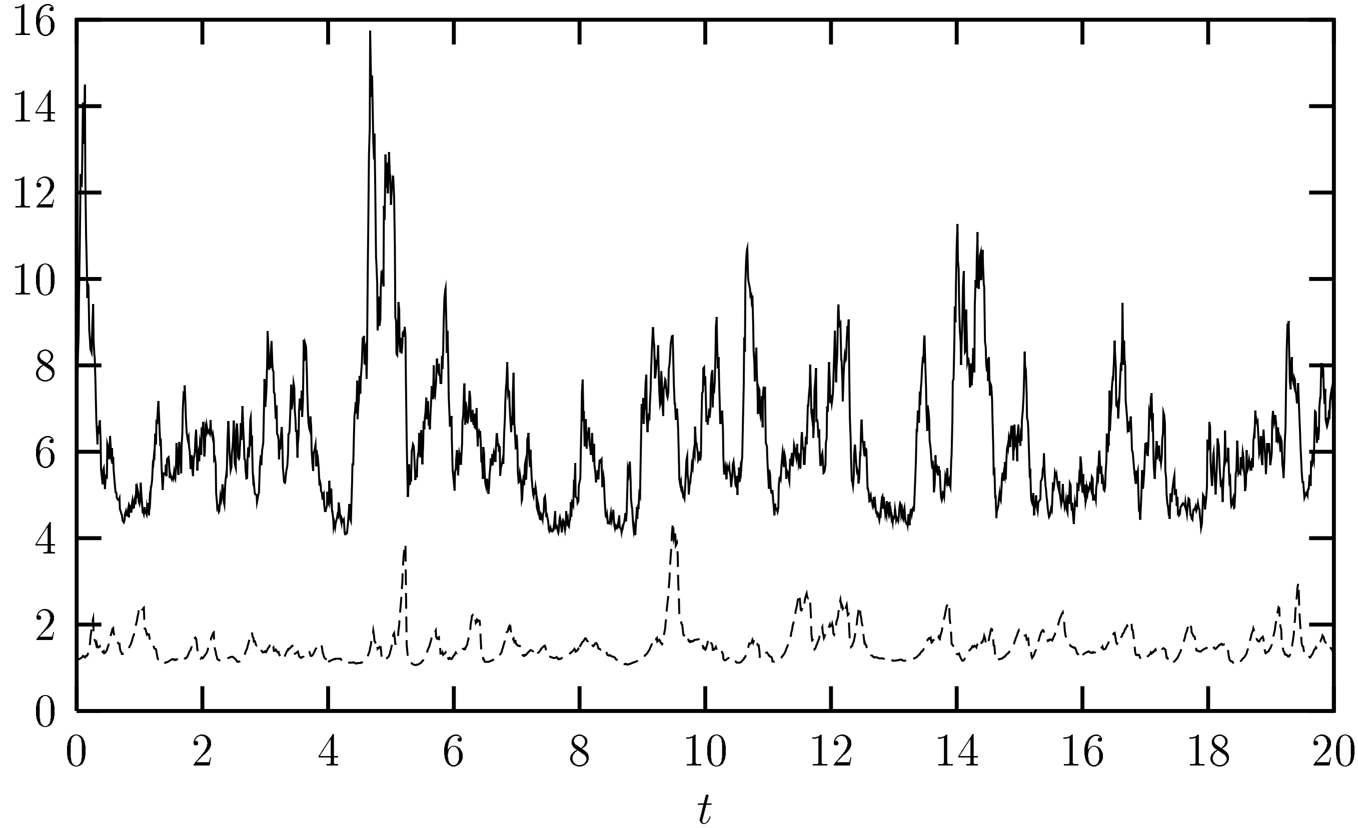

FIG. 3. Typical time evolution of the total velocity energy $E_u(t)$ (solid) and total passive energy $E_w(t)$ (dashed) (case A). Curves have been shifted along the y-axis for the sake of clarity

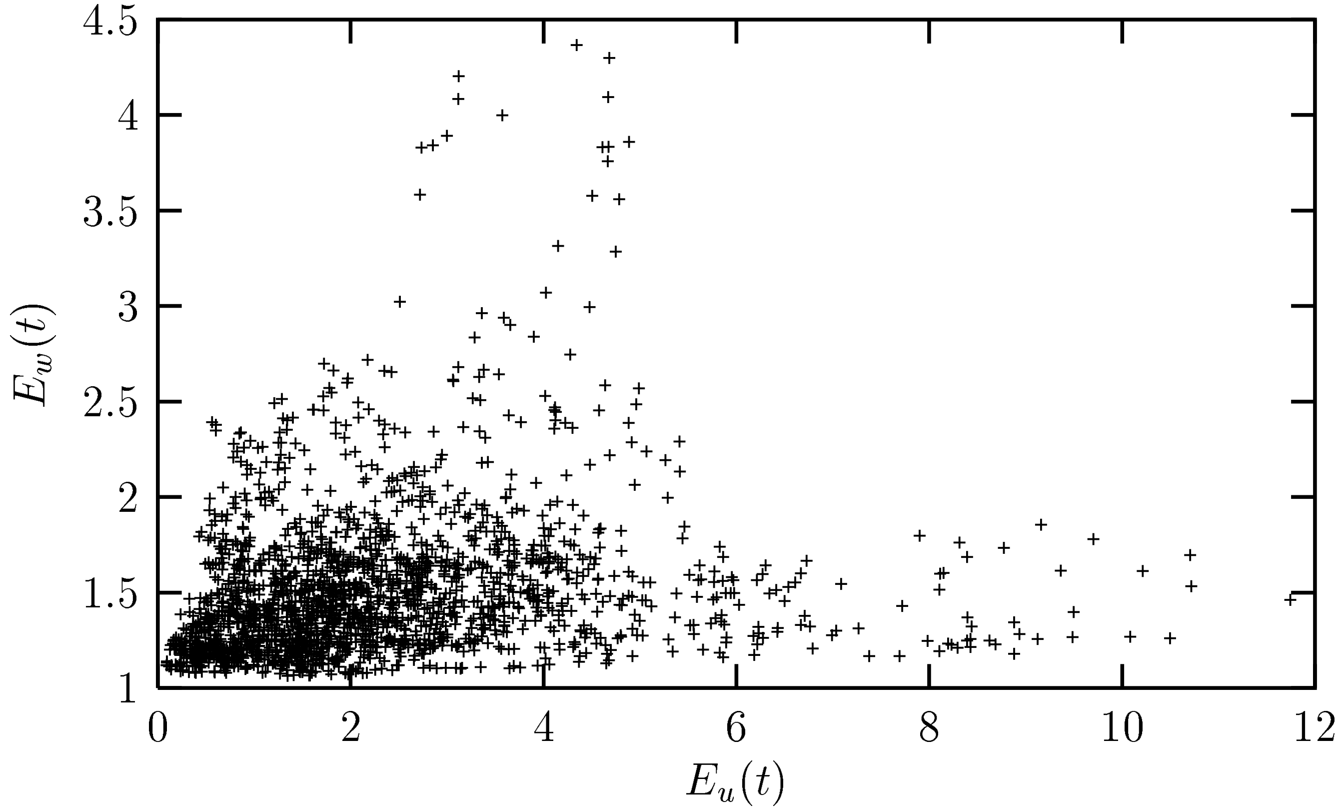

FIG. 4. Scatter plot of $E_u(t)$ versus $E_w(t)$ with forcing of case (A). Notice that the two fields have not perfect correlation.

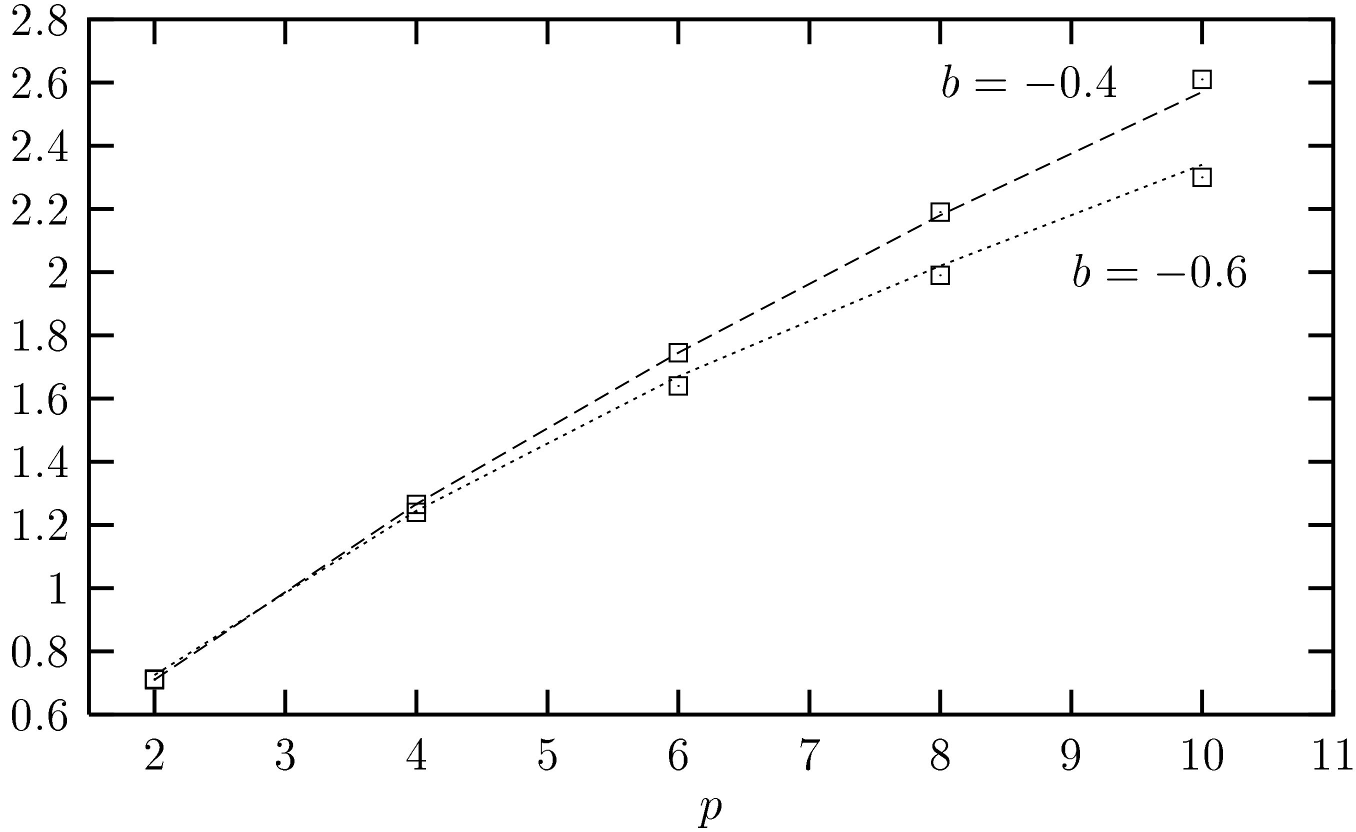

FIG. 5. Scaling exponents of velocity flux and passive flux for two different values of the parameter b in the "Sabra" shell model equations (11). □ corresponds to the velocity flux exponents, $\zeta(p/3)$, and the dashed curves to the passive flux exponents, $\sigma(p)$. Above: the case with $b = -0.4$, below the case with $b = -0.6$. Notice that despite the remarkable change in the intermittent properties the fluxes follow eachother perfectly.

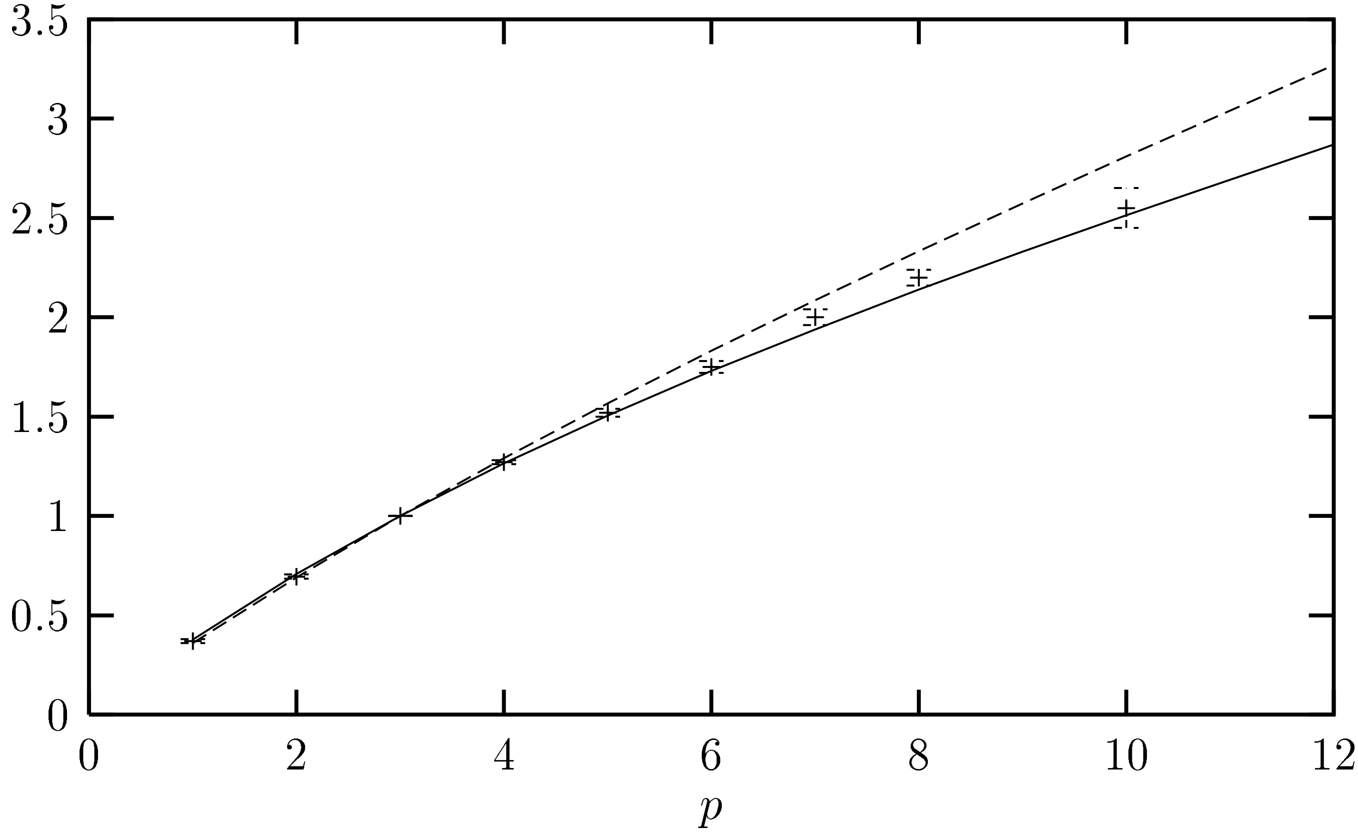

FIG. 6. Comparison between the scaling exponents computed by eqn. (23) using: numerical results for the passive scalar [17] (solid curve); the Kraichan formula for the passive scalar [1] (dashed curve). Experimental data (+) are given with their error bars.